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Multiple solutions for elliptic systems with nonlinearities of arbitrary growth

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Abstract

We prove the existence of infinitely many solutions for symmetric elliptic systems with nonlinearities of arbitrary growth. Moreover, if the symmetry of the problem is broken by a small enough perturbation term, we find at least three solutions. The proofs utilise a variational setting given by de Figueiredo and Ruf in order to prove an existence's result and the "algebraic" approach based on the Pohozaev's fibering method.

MSC: primary 35J55; secondary 35J50, 35J60, 47J30

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1. Introduction

Let us consider the following system of equations

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$. It is already known (see [2,3,8]) that in the "model case"

$$f(s) = s^{q-1}, \qquad g(s) = s^{p-1}, \quad q, p > 2$$

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(here and in the following, $s^\alpha = \operatorname{sgn}(s)|s|^\alpha$) system (1.1) has a nontrivial solution provided that

$$1 > \frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}. \tag{1.2}$$

If $N = 2$ this condition holds for any $p > 2$ and $q > 2$. Really, by using the Trudinger–Moser inequality, existence of solutions of (1.1) has been proved in [4] even if f and g grow more than polynomially.

If $N \geqslant 3$, the curve of $(p, q) \in \mathbf{R}^2$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 - \frac{2}{N}$ is the so-called "critical hyperbola": in this case, indeed, because of the lack of compactness of the problem, non-existence of solutions has been proved in [9] and [17] using Pohozaev type arguments. Existence of solutions of (1.1) has been proved in [7] even in the case $0 < (p-1)(q-1) < 1$. Clearly, the cases $q, p > 2$, $\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}$ or $0 < (p-1)(q-1) < 1$ do not cover the whole region below the critical hyperbola. However, more recently de Figueiredo and Ruf in [6] show that a nontrivial solution of system (1.1) exists if $g(s) = s^{p-1}$ with $1 < p < \frac{N}{N-2}$ and f is superlinear with $q > 2$ if $p > 2$ or $q > 1 + \frac{1}{p-1}$ if $p \leqslant 2$, without growth restrictions on the function f. In the border-line case $p = \frac{N}{N-2}$ a critical growth of exponential type for f has been obtained in [14].

The aim of this paper is to state a multiplicity result in the case $1 < p \leqslant 2$. Indeed, we will consider a sublinear nonlinearity in the form of power in one equation and a superlinear nonlinearity in the other equation and we will prove that, if f is odd, under the same growth conditions for f and g introduced in [6] system (1.1) has infinitely many pairs of solutions. A different proof of the de Figueiredo and Ruf's existence result will be also given, always in the case $1 < p \leqslant 2$. Finally, if f has the form of power too, multiple solutions will be found even if the symmetry of the problem is broken by a perturbation term in the second equation.

More precisely, let $f \in C(\mathbf{R})$ and, setting $F(s) = \int_0^s f(t)\,dt$, assume

(f_1) there exist constants $\theta > 1 + \frac{1}{p-1}$ and $s_0 \geqslant 0$ such that

$$0 < \theta F(s) \leqslant f(s)s \quad \text{for all } |s| \geqslant s_0;$$

(f_2) $f(s) = o(s^{\frac{1}{p-1}})$ for s near 0.

We will prove that by (f_1) and (f_2) it follows that for any $w \in S$, S unit sphere in a suitable Banach space, the equation

$$|\lambda|^{\frac{p}{p-1}} - \int_\Omega f(\lambda w)\lambda w\,dx = 0$$

in $\lambda \in \mathbf{R}$ has at least two opposite sign solutions $\lambda_-(w) < 0 < \lambda_+(w)$. Assume that

(R) there exist two selections $\lambda_\pm(w) \in C^1(S)$.

Let us point out that the regularity assumption (R) is verified e.g. if $f(s) = s^{q-1}$ (see Section 2).

We will prove the following results.

Theorem 1.1. *Let $1 < p \leqslant 2$ if $N = 2, 3$ or $1 < p < \frac{N}{N-2}$ if $N \geqslant 4$. Assume that f verifies (f_1), (f_2) and (R).*

Then, the system

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has a nontrivial solution. If f satisfies the additional condition

(f_3) *f is odd,*

then, (1.3) *has infinitely many pairs of solutions.*

Remark 1.2. As already remarked in [6], it is surprising that, if one nonlinearity, say g, has polynomial growth, then no growth condition is needed for the other nonlinearity f other than the Ambrosetti–Rabinowitz condition (f_1); on the contrary, for the single equation $-\Delta u = f(u)$ growth conditions are in general necessary in order to state existence of solutions: for non-existence results in this case see [5] if $N = 2$ and [10] if $N \geqslant 3$.

Remark 1.3. It is well known that by (f_1) a positive constant c_1 exists such that

$$F(s) \geqslant c_1|s|^\theta \quad \text{for all } |s| \geqslant s_0.$$

For completeness, we also prove the following result:

Theorem 1.4. *Let $N \geqslant 4$, $\frac{N}{N-2} \leqslant p \leqslant 2$ and q such that $(p, q) \in \mathbf{R}^2$ satisfies* (1.2). *Assume that f verifies (f_1), (f_2), (R) and*

(f_4) *there exist two positive constants a_1, a_2 such that $|f(s)| \leqslant a_1|s|^{q-1} + a_2$.*

Then, (1.3) *has a nontrivial solution. Moreover, if f is odd,* (1.3) *has infinitely many pairs of solutions.*

Remark 1.5. By (f_1), Remark 1.3 and (f_4) it follows that it is $\theta \leqslant q$ which is compatible with $\theta > \frac{p}{p-1}$ because $1 > \frac{1}{p} + \frac{1}{q}$. Obviously, in this case only a polynomial growth is allowed for f; moreover, if $\frac{N}{N-2} < p \leqslant 2$ assumption $\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}$ implies $q < \frac{Np}{N(p-1)-2p}$ while if $p = \frac{N}{N-2}$ there is no upper bound for q.

On the contrary, if p is as in Theorem 1.1 no polynomial growth is needed for f.

Now, let us consider the following nonhomogeneous system

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = u^{q-1} + h(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where p is chosen as in Theorem 1.1 or 1.4.

Clearly, if $h(x)=0$, fixed $q>1+\frac{1}{p-1}$ the function $f(u)=u^{q-1}$ satisfies (f_1) (with $\theta=q$), (f_2) and (f_4); moreover, as already remarked, also condition (R) holds, hence, Theorems 1.1 and 1.4 imply the existence of infinitely many solutions of system (1.4). On the other hand, if $h(x)\neq 0$, the nonlinear term $f(x,u)=u^{q-1}+h(x)$ does not verify (f_2), then even the de Figueiredo and Ruf existence's result cannot apply. Equations and systems with perturbed symmetry have been widely studied, see e.g. [1] and [15]. For systems of type (1.4) we refer in particular to [16], where a curve below the critical hyperbola was found below which there exist infinitely many solutions for any perturbation h in $L^2(\Omega)$. Here, we state the following different multiplicity result.

Theorem 1.6. *Let $1<p\leqslant 2$ if $N=2,3$ or $1<p<\frac{N}{N-2}$ if $N\geqslant 4$ and $q>1+\frac{1}{p-1}$. Then, for any $h\in L^1(\Omega)$ with $\|h\|_1$ small enough system* (1.4) *has at least three solutions $\bar{u}_i$ such that*

$$\int_\Omega h\bar{u}_1\,dx\leqslant 0,\qquad \int_\Omega h\bar{u}_2\,dx\geqslant 0\quad \textit{and}\quad \int_\Omega h\bar{u}_3\,dx\geqslant 0.$$

If $\frac{N}{N-2}\leqslant p\leqslant 2$ $(N\geqslant 4)$ and q is such that $1>\frac{1}{p}+\frac{1}{q}>1-\frac{2}{N}$, the same result holds for any $h\in L^\gamma(\Omega)$ with $\|h\|_\gamma$ small enough, where $\gamma=\frac{Np}{N+2p}$.

Remark 1.7. Let us point out that in the border-line case $p=\frac{N}{N-2}$ we obtain that for any $q>\frac{N}{2}$ system (1.4) has at least three solutions for any $h\in L^1(\Omega)$ with $\|h\|_1$ small enough.

Our existence and multiplicity results will be obtained with the same variational setting given in [6] if $p\leqslant 2$ by using the "algebraic" approach based on the fibering method introduced by Pohozaev (see [11–13]). Unfortunately, this method seems do not work in the case $2<p<\frac{N}{N-2}$ $(N=2,3)$ for which the existence of one solution has been proved in [6].

2. Proof of Theorem 1.1

Let $g(s)=s^{p-1}$ with $1<p\leqslant 2$ if $N=2,3$ or $1<p<\frac{N}{N-2}$ if $N\geqslant 4$ and f verifying (f_1) and (f_2). In general, setting $F(s)=\int_0^s f(t)\,dt$ and $G(s)=\int_0^s g(t)\,dt$, the natural functional associated to the general system (1.1) is

$$J(u,v)=\int_\Omega \nabla u\nabla v\,dx-\int_\Omega \big(F(u)+G(v)\big)\,dx.$$

Clearly, J is well defined and of class C^1 on the Sobolev space $H_0^1(\Omega)\times H_0^1(\Omega)$ if each of the nonlinear terms is subcritical. However, we are interested to a different type of assumptions, since we want to prove that, if $G(s)=\frac{s^p}{p}$, no growth limitation for $F(s)$ is needed. As $1<p\leqslant 2$, system (1.3) becomes

$$\begin{cases} (-\Delta u)^{\frac{1}{p-1}}=v & \text{in } \Omega,\\ -\Delta v=f(u) & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega,\end{cases}$$

which is equivalent to the single equation

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} = f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

In order to give a suitable variational formulation of the problem, first of all let us consider the space $E = W^{2,\frac{p}{p-1}}(\Omega) \cap W_0^{1,\frac{p}{p-1}}(\Omega)$ endowed with the norm

$$\|u\|_E = \left(\int_\Omega |\Delta u|^{\frac{p}{p-1}}\, dx \right)^{\frac{p-1}{p}}$$

equivalent to the usual intersection norm

$$\|u\| = \max\left\{ \|u\|_{W^{2,\frac{p}{p-1}}}, \|u\|_{W_0^{1,\frac{p}{p-1}}} \right\}.$$

The following result will be stated.

Proposition 2.1. *The weak solutions of* (2.1) *are the critical points of the energy functional*

$$I(u) = \frac{p-1}{p} \int_\Omega |\Delta u|^{\frac{p}{p-1}}\, dx - \int_\Omega F(u)\, dx, \quad u \in E.$$

Proof. It is enough to point out that I is well defined and of class C^1 on the space E and its Fréchet differential has the form

$$I'(u)[h] = \int_\Omega (-\Delta u)^{\frac{1}{p-1}} (-\Delta h)\, dx - \int_\Omega f(u) h\, dx, \quad u, h \in E.$$

In order to prove that no growth condition on f is needed for the regularity of the term $\varphi(u) = \int_\Omega F(u)\, dx$, let us recall that, since $p < \frac{N}{N-2}$, it is $\frac{p}{p-1} > 1 + \frac{N-2}{2} = \frac{N}{2}$, then the compact imbedding

$$W^{2,\frac{p}{p-1}}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega) \tag{2.2}$$

holds, where $C_B(\Omega)$ is the space of the continuous bounded functions on Ω; hence, the C^1 regularity of $F(s)$ implies that the function φ is well defined and of class C^1 on the space E with $\varphi'(u)[h] = \int_\Omega f(u) h\, dx$. $\square$

Following Pohozaev's fibering method (see [11–13]), we look for critical points of I in the form $u = \lambda w$, where $(\lambda, w) \in \mathbf{R} \times E$, $(\lambda, w) \neq (0, 0)$ and w verifies a condition of the type $H(\lambda, w) = c \neq 0$ with H suitable fibering functional. In this case we use the so-called spherical fibering, that is we take $H(\lambda, w) = \|w\|_E$. Hence, we find solutions $u = \lambda w$ with $\lambda \in \mathbf{R}$ and $w \in S$,

$$S = \left\{ w \in E \colon \|w\|_E = 1 \right\}.$$

Therefore, we associate with I a functional $\tilde{I}$ defined on $\mathbf{R} \times E$ by

$$\tilde{I}(\lambda, w) = I(\lambda w) = \frac{p-1}{p} |\lambda|^{\frac{p}{p-1}} \int_\Omega |\Delta w|^{\frac{p}{p-1}}\, dx - \int_\Omega F(\lambda w)\, dx.$$

Clearly, the restriction of $\tilde{I}$ on $\mathbf{R} \times S$, still denoted by $\tilde{I}$, becomes

$$\tilde{I}(\lambda, w) = \frac{p-1}{p} |\lambda|^{\frac{p}{p-1}} - \int_\Omega F(\lambda w)\, dx.$$

It can be proved that if $(\lambda, w) \in (\mathbf{R}/\{0\}) \times S$ is a conditionally stationary point of the functional $\tilde{I}$ on $\mathbf{R} \times S$, then the vector $u = \lambda w$ is a nonzero "free" stationary point of the functional I, that is, $I'(u) = 0$. In other words, any critical point (λ, w) of $\tilde{I}$ restricted on $(\mathbf{R}/\{0\}) \times S$ generates the free nontrivial critical point $u = \lambda w$ of I and vice versa, that is, the equation

$$I'(u) = 0, \quad u \neq 0$$

is equivalent to the system

$$\begin{cases} \dfrac{\partial \tilde{I}}{\partial \lambda}(\lambda, w) = 0, \\ \dfrac{\partial \tilde{I}}{\partial w}(\lambda, w) = 0 \end{cases}$$

for $w \in S$.

In the following we will call the first scalar equation of the previous system the "bifurcation equation."

In our case, the bifurcation equation $\frac{\partial \tilde{I}}{\partial \lambda}(\lambda, w) = 0$ takes the form

$$|\lambda|^{\frac{p}{p-1}-2}\lambda - \int_\Omega f(\lambda w) w\, dx = 0$$

or equivalently, for $\lambda \neq 0$,

$$|\lambda|^{\frac{p}{p-1}} - \int_\Omega f(\lambda w)\lambda w\, dx = 0. \tag{2.3}$$

Obviously, in the particular case $f(s) = s^{q-1}$ straight calculations imply that Eq. (2.3) has exactly two solutions $\lambda_\pm(w) = \pm(\int_\Omega |w|^q)^{-t}$, $t = q(q - \frac{p}{p-1})$; moreover, $\lambda_\pm(w) \in C^1(S)$. Our aim is to prove that also in the general case there exist at least two nontrivial opposite sign solutions of (2.3).

Setting

$$\varphi_w(\lambda) = 1 - |\lambda|^{-\frac{p}{p-1}} \int_\Omega f(\lambda w)\lambda w\, dx,$$

the following lemma holds.

Lemma 2.2. *Assume that f satisfies (f_1) and (f_2). Then, for all $w \in S$ it is*

(i) $$\lim_{\lambda \to 0} \varphi_w(\lambda) = 1,$$

(ii) $$\lim_{|\lambda| \to +\infty} \varphi_w(\lambda) = -\infty.$$

Proof. Fixed $w \in S$, by (2.2) w is a nontrivial bounded continuous function, then there exists $M > 0$ such that for all $x \in \Omega$ it is $|w(x)| \leqslant M$.

Now, by (f_2) for any $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|s| \leqslant \delta$ implies $\frac{|f(s)|}{|s|^{\frac{1}{p-1}}} < \varepsilon$. Taken λ small enough, $|\lambda| < \frac{\delta}{M}$, it results

$$\frac{|f(\lambda w(x))|}{|\lambda w(x)|^{\frac{1}{p-1}}} < \varepsilon \quad \text{for any } x \in \Omega,$$

hence,

$$\lim_{\lambda \to 0} |\lambda|^{-\frac{p}{p-1}} \int_\Omega f(\lambda w)\lambda w \, dx = 0,$$

so (i) follows. On the other hand, we can write

$$|\lambda|^{-\frac{p}{p-1}} \int_\Omega f(\lambda w)\lambda w \, dx = |\lambda|^{-\frac{p}{p-1}} \left\{ \int_{\Omega_\lambda^-} f(\lambda w)\lambda w \, dx + \int_{\Omega_\lambda^+} f(\lambda w)\lambda w \, dx \right\}, \tag{2.4}$$

where $\Omega_\lambda^- = \{x \in \Omega\colon |\lambda w(x)| < s_0\}$ and $\Omega_\lambda^+ = \{x \in \Omega\colon |\lambda w(x)| \geqslant s_0\}$, s_0 being the positive constant introduced in (f_1). Clearly, the boundedness of $f(\lambda w)\lambda w$ on Ω_λ^- implies that

$$\lim_{|\lambda| \to +\infty} |\lambda|^{-\frac{p}{p-1}} \int_{\Omega_\lambda^-} f(\lambda w)\lambda w \, dx = 0 \tag{2.5}$$

while by (f_1) and Remark 1.3 it follows that

$$|\lambda|^{-\frac{p}{p-1}} \int_{\Omega_\lambda^+} f(\lambda w)\lambda w \, dx \geqslant \theta |\lambda|^{-\frac{p}{p-1}} \int_{\Omega_\lambda^+} F(\lambda w) \, dx \geqslant c_1 \theta |\lambda|^{-\frac{p}{p-1}} \int_{\Omega_\lambda^+} |\lambda w|^\theta \, dx.$$

Denoted by λ^* a real positive number such that $\Omega_{\lambda^*}^+ \neq \emptyset$, for $|\lambda| \geqslant \lambda^*$ it results $\Omega_{\lambda^*}^+ \subset \Omega_\lambda^+$ and therefore a positive constant $c_2 > 0$ exists such that for $|\lambda| \geqslant \lambda^*$ it is

$$|\lambda|^{-\frac{p}{p-1}} \int_{\Omega_\lambda^+} f(\lambda w)\lambda w \, dx \geqslant c_1 \theta |\lambda|^{\theta - \frac{p}{p-1}} \int_{\Omega_{\lambda^*}^+} |w|^\theta \, dx \geqslant c_2 |\lambda|^{\theta - \frac{p}{p-1}}.$$

As $\theta > \frac{p}{p-1}$, we conclude that

$$\lim_{|\lambda|\to+\infty} |\lambda|^{-\frac{p}{p-1}} \int\limits_{\Omega_\lambda^+} f(\lambda w)\lambda w\,dx = +\infty, \tag{2.6}$$

hence, (ii) follows by (2.4)–(2.6). □

From the previous lemma, analyzing the graph of φ_w it follows that for any $w \in S$ Eq. (2.3) has at least two nontrivial opposite sign solutions. By regularity assumption (R) there exist two selections $\lambda_\pm(w)$ which are of class C^1 on S. Then, the C^1 functional $\hat{I}_\pm(w) = \tilde{I}(\lambda_\pm(w), w)$ becomes

$$\hat{I}_\pm(w) = \frac{p-1}{p} \int\limits_\Omega f\big(\lambda_\pm(w)w\big)\lambda_\pm(w)w\,dx - \int\limits_\Omega F\big(\lambda_\pm(w)w\big)\,dx.$$

Now, we are interested to find critical points of $\hat{I}_\pm$ on S. From now on, we deal with the functional $\hat{I}_+$; in a similar way, it is possible to conclude for the functional $\hat{I}_-$.

It is easy to prove that by (f_1) the functional $\hat{I}_+$ is bounded from below. In fact, if $\Omega_{\lambda_+}^+$ and $\Omega_{\lambda_+}^-$ are defined as in the previous lemma, by (f_1) a real constant k exists such that it results

$$\begin{aligned}
\hat{I}_+(w) &= \int\limits_\Omega \left(\frac{p-1}{p} f\big(\lambda_+(w)w\big)\lambda_+(w)w - F\big(\lambda_+(w)w\big)\right)dx \\
&= \int\limits_{\Omega_{\lambda_+}^-} \left(\frac{p-1}{p} f\big(\lambda_+(w)w\big)\lambda_+(w)w - F\big(\lambda_+(w)w\big)\right)dx \\
&\quad + \int\limits_{\Omega_{\lambda_+}^+} \left(\frac{p-1}{p} f\big(\lambda_+(w)w\big)\lambda_+(w)w - F\big(\lambda_+(w)w\big)\right)dx \\
&\geqslant k + \left(\frac{p-1}{p}\theta - 1\right) \int\limits_{\Omega_{\lambda_+}^+} F\big(\lambda_+(w)w\big)\,dx \geqslant k,
\end{aligned}$$

that is, $\hat{I}_+$ is bounded from below.

We study the following constrained variational problem: find a minimizer $w_+ \in E$ of the problem

$$m_+ = \inf\big\{\hat{I}_+(w):\ w \in S\big\}. \tag{2.7}$$

To this aim, we prove the following result.

Lemma 2.3. *Assume that (f_1) and (f_2) hold. If $\{w_n\} \subset S$ is such that the sequence $\{\hat{I}_+(w_n)\}$ is bounded from above, then the corresponding sequence $\{\lambda_+(w_n)\}$ is bounded and bounded away from zero.*

Proof. Let $\{w_n\} \subset S$ such that $\{\hat{I}_+(w_n)\}$ is bounded from above. By (2.2), up to subsequence,

$$w_n \rightharpoonup w \quad \text{weakly in } E \text{ and uniformly in } \Omega. \tag{2.8}$$

For simplicity, denote by $\{\lambda_n\}$ the sequence $\{\lambda_+(w_n)\}$. Our aim is to find two strictly positive constants c_3, c_4 such that

$$c_3 \leqslant \lambda_n \leqslant c_4 \quad \text{for any integer } n. \tag{2.9}$$

In order to establish the first inequality in (2.9), assume by contradiction that, up to subsequence, $\lambda_n \to 0$. By (2.8) $\lambda_n w_n \to 0$ uniformly in Ω, then, arguing as in the proof of (i) of the previous lemma, by (f_2) it follows that

$$\lim_n (\lambda_n)^{-\frac{p}{p-1}} \int_\Omega f(\lambda_n w_n)\lambda_n w_n \, dx = 0,$$

hence, by (2.3) we easily obtain the contradiction.

Now, in order to show the second inequality in (2.9), assume by contradiction that $\lambda_n \to +\infty$ (passing to a subsequence). Let

$$\Omega_n^- = \{x \in \Omega\colon |\lambda_n w_n(x)| < s_0\} \quad \text{and} \quad \Omega_n^+ = \{x \in \Omega\colon |\lambda_n w_n(x)| \geqslant s_0\}.$$

Then by (2.3) for any n it is

$$(\lambda_n)^{-\frac{p}{p-1}} \int_{\Omega_n^-} f(\lambda_n w_n)\lambda_n w_n \, dx + (\lambda_n)^{-\frac{p}{p-1}} \int_{\Omega_n^+} f(\lambda_n w_n)\lambda_n w_n \, dx = 1. \tag{2.10}$$

Since the sequence $\{\lambda_n w_n\}$ is uniformly bounded on the set Ω_n^-, we have

$$\lim_n (\lambda_n)^{-\frac{p}{p-1}} \int_{\Omega_n^-} f(\lambda_n w_n)\lambda_n w_n \, dx = 0 \tag{2.11}$$

and

$$\left\{ \int_{\Omega_n^-} F(\lambda_n w_n) \, dx \right\} \text{ is bounded.} \tag{2.12}$$

In particular, by (2.10) and (2.11) it follows that

$$\lim_n (\lambda_n)^{-\frac{p}{p-1}} \int_{\Omega_n^+} f(\lambda_n w_n)\lambda_n w_n \, dx = 1.$$

On the other hand it is

$$\int_{\Omega_n^+} f(\lambda_n w_n)\lambda_n w_n \, dx \geqslant \theta \int_{\Omega_n^+} F(\lambda_n w_n) \, dx,$$

then, passing to a subsequence we can consider

$$\lim_n (\lambda_n)^{-\frac{p}{p-1}} \int_{\Omega_n^+} F(\lambda_n w_n)\,dx = l, \quad 0 \leqslant l \leqslant \frac{1}{\theta}. \tag{2.13}$$

Now, let us point out that

$$\hat{I}_+(w_n) = (\lambda_n)^{\frac{p}{p-1}} \left(\frac{p-1}{p} - (\lambda_n)^{-\frac{p}{p-1}} \int_{\Omega_n^+} F(\lambda_n w_n)\,dx \right) - \int_{\Omega_{\lambda_n}^-} F(\lambda_n w_n)\,dx,$$

hence, by $\theta > \frac{p}{p-1}$, (2.12) and (2.13) we obtain

$$\lim_n \hat{I}_+(w_n) = +\infty,$$

which contradicts the boundedness of $\{\hat{I}_+(w_n)\}$. $\square$

Let $\{w_n\} \subset S$ be a minimizing sequence of the problem (2.7), i.e.,

$$w_n \in E, \qquad \|w_n\|_E = 1, \qquad \hat{I}_+(w_n) \to m_+.$$

Then, there exists $\bar{w}_+ \in E$ with $\|\bar{w}_+\|_E \leqslant 1$ such that, up to subsequence, (2.8) holds. By virtue of Lemma 2.3 the corresponding sequence $\{\lambda_+(w_n)\}$ converges, up to subsequence, to a real number $\bar{\lambda}_+$ different from 0; clearly by (2.8), passing to the limit in the bifurcation equation, $(\bar{\lambda}_+, \bar{w}_+)$ still solves (2.3), that is $\bar{\lambda}_+ = \lambda_+(\bar{w}_+)$ with $\bar{w}_+ \neq 0$. In other words,

$$\lambda_+(w_n) \to \lambda_+(\bar{w}_+),$$

therefore, always by (2.8), we conclude that

$$\hat{I}_+(w_n) \to \hat{I}_+(\bar{w}_+) = m_+.$$

In order to prove that $\|\bar{w}_+\|_E = 1$, let us remark that, for any $\theta > 0$, by (2.3) it is

$$\begin{aligned}
\frac{d}{d\theta}\hat{I}_+(\theta\bar{w}_+) &= \frac{d}{d\theta}\left[\frac{p-1}{p}\big(\lambda_+(\theta\bar{w}_+)\big)^{\frac{p}{p-1}} - \int_\Omega F\big(\lambda_+(\theta\bar{w}_+)\theta\bar{w}_+\big)\,dx \right] \\
&= \left[\big(\lambda_+(\theta\bar{w}_+)\big)^{\frac{1}{p-1}} - \int_\Omega f\big(\lambda_+(\theta\bar{w}_+)\theta\bar{w}_+\big)\theta\bar{w}_+\,dx \right] \\
&\quad \times \frac{d}{d\theta}\lambda_+(\theta\bar{w}_+)\bar{w}_+ - \int_\Omega f\big(\lambda_+(\theta\bar{w}_+)\theta\bar{w}_+\big)\lambda_+(\theta\bar{w}_+)\bar{w}_+\,dx \\
&= -\frac{1}{\theta}\big(\lambda_+(\theta\bar{w}_+)\big)^{\frac{p}{p-1}} < 0,
\end{aligned}$$

then $\hat{I}_+(\theta\bar{w}_+)$ decreases with respect to $\theta \in]0,1]$ and attains its minimum for $\theta = 1$, i.e. $\bar{w}_+ \in S$. Arguing in a similar way, we can prove that also the functional $\hat{I}_-$ attains its minimum on S at a point $\bar{w}_-$.

According to the fibering method, we conclude that

$$\bar{u}_+ = \lambda_+(\bar{w}_+)\bar{w}_+ \quad \text{and} \quad \bar{u}_- = \lambda_-(\bar{w}_-)\bar{w}_-$$

are two nontrivial critical points of I, hence, by Proposition 2.1, two nontrivial solutions of system (1.3). In general, these solutions can be equal. On the contrary, if f is odd, it is $\lambda_+(\bar{w}_+) = -\lambda_-(\bar{w}_-)$, then $\hat{I} = \hat{I}_+ = \hat{I}_-$ and $\bar{w}_+ = \bar{w}_-$, hence $\bar{u}_+ = -\bar{u}_-$ are two opposite sign solutions of system (1.3). Moreover, $\hat{I}$ is even, bounded from below, of class C^1 and "almost weakly continuous" on S, that is, for any $\{w_n\} \subset S$ such that $w_n \rightharpoonup w$ and $\{\hat{I}(w_n)\}$ is bounded from above, it is $\hat{I}(w_n) \to \hat{I}(w)$. Then, by applying the classical Lusternik–Schnirelmann theory we prove that $\hat{I}$ has a sequence of geometrical different conditionally critical points $w_1, w_2, \dots, w_n, \dots$ on S with $\hat{I}(w_n) \to +\infty$ as $n \to \infty$. Hence, by the fibering method I has a sequence of geometrically different critical points $\pm u_1, \pm u_2, \dots, \pm u_n, \dots$ with $u_n(x) = \lambda(w_n)w_n$ such that $I(u_n) \to +\infty$, so the conclusion of the proof of Theorem 1.1 follows.

3. Proof of Theorem 1.4

Let $N \geqslant 4$ and $(p,q) \in \mathbf{R}^2$ such that $1 - \frac{2}{N} < \frac{1}{p} + \frac{1}{q} < 1$ with $\frac{N}{N-2} \leqslant p \leqslant 2$. Then, it is

$$W^{2,\frac{p}{p-1}}(\Omega) \hookrightarrow L^{\frac{Np}{N(p-1)-2p}}(\Omega).$$

Let us point out that the exponent $\frac{Np}{N(p-1)-2p}$ satisfies the condition

$$\frac{1}{p} + \frac{1}{\frac{Np}{N(p-1)-2p}} = 1 - \frac{2}{N},$$

i.e. we are on the critical hyperbola. Hence, for $q < \frac{Np}{N(p-1)-2p}$ we are below the hyperbola, and we have

$$E \hookrightarrow\hookrightarrow L^q \quad \text{compactly}. \tag{3.1}$$

Hence, also in this case functional I is well defined and C^1 on the space E and Proposition 2.1 holds. According to the fibering method, we look for critical points $u = \lambda w$ of I, $w \in S$. Since in this case the function w can be unbounded, some modifications are needed with respect to the arguments of the previous section.

To this aim, we prove the following result.

Lemma 3.1. *Assume that f satisfies (f_2) and (f_4). Then, for any $\varepsilon > 0$ two positive constants c_5, c_6 exist such that for any $(\lambda, w) \in \mathbf{R} \times S$ it is*

$$\left|\lambda^{-\frac{p}{p-1}} \int_\Omega f(\lambda w)\lambda w \, dx\right| \leqslant c_5\varepsilon + c_6\lambda^{q-\frac{p}{p-1}}.$$

Proof. Fix $w \in S$. By (f_2) it follows that for any $\varepsilon > 0$ there exists $\delta > 0$ s.t. $|s| \leqslant \delta$ implies $\frac{|f(s)|}{|s|^{\frac{1}{p-1}}} < \varepsilon$. Now, we can write

$$\lambda^{-\frac{p}{p-1}} \int_{\Omega} f(\lambda w)\lambda w\,dx = \lambda^{-\frac{p}{p-1}} \left\{ \int_{\Omega^-_{\lambda,\delta}} f(\lambda w)\lambda w\,dx + \int_{\Omega^+_{\lambda,\delta}} f(\lambda w)\lambda w\,dx \right\}, \tag{3.2}$$

where $\Omega^-_{\lambda,\delta} = \{x \in \Omega\colon |\lambda w(x)| < \delta\}$ and $\Omega^+_{\lambda,\delta} = \{x \in \Omega\colon |\lambda w(x)| \geqslant \delta\}$.

Clearly, it is

$$\left| \lambda^{-\frac{p}{p-1}} \int_{\Omega^-_{\lambda,\delta}} f(\lambda w)\lambda w\,dx \right| \leqslant \int_{\Omega^-_{\lambda,\delta}} \frac{|f(\lambda w)|}{|\lambda w|^{\frac{1}{p-1}}} |w|^{\frac{p}{p-1}}\,dx \leqslant \varepsilon \int_{\Omega} |w|^{\frac{p}{p-1}}\,dx,$$

so, by (3.1) denoted by $C_{\frac{p}{p-1}}$ the imbedding constant of E in $L^{\frac{p}{p-1}}$, we obtain

$$\left| \lambda^{-\frac{p}{p-1}} \int_{\Omega^-_{\lambda,\delta}} f(\lambda w)\lambda w\,dx \right| \leqslant \varepsilon (C_{\frac{p}{p-1}})^{\frac{p}{p-1}} \leqslant \varepsilon c_5. \tag{3.3}$$

On the other hand, by (f_4) we can fix K large enough such that it results

$$\left| f(s) \right| \leqslant K|s|^{q-1} \quad \text{for all } |s| \geqslant \left(\frac{a_2}{K - a_1} \right)^{\frac{1}{q-1}} \text{ with } \left(\frac{a_2}{K - a_1} \right)^{\frac{1}{q-1}} < \delta,$$

and therefore, using again (3.1), it is

$$\left| \lambda^{-\frac{p}{p-1}} \int_{\Omega^+_{\lambda,\delta}} f(\lambda w)\lambda w\,dx \right| \leqslant K\lambda^{q-\frac{p}{p-1}} \int_{\Omega^+_{\lambda,\delta}} \left| w(x) \right|^q dx \leqslant K\lambda^{q-\frac{p}{p-1}} C_q, \tag{3.4}$$

where C_q is the constant of the imbedding of E in L^q. By (3.2)–(3.4) we obtain the conclusion. □

Let us point out that the constant c_5 introduced in Lemma 3.1 is independent of ε as ε tends to 0. Then, as $q > \frac{p}{p-1}$, Lemma 3.1 allows us to prove part (i) in Lemma 2.2 and the first part of Lemma 2.3. All the remainder of the proof follows as in the previous section.

4. The perturbation case

Let us consider the case $h \neq 0$. Thus, the solutions of system (1.4) are the critical points of the functional

$$I_h(u) = \frac{p-1}{p} \int_{\Omega} |\Delta u|^{\frac{p}{p-1}}\,dx - \frac{1}{q} \int_{\Omega} |u|^q\,dx - \int_{\Omega} hu\,dx$$

on the Banach space E defined as in Sections 2 or 3 according to the choice of p and q. Following the notations introduced in the previous sections, let us denote by $\tilde{I}_h$ both the extension of I_h to the space $\mathbf{R} \times E$, that is

$$\tilde{I}_h(\lambda, w) = I_h(\lambda w) = \frac{p-1}{p}|\lambda|^{\frac{p}{p-1}} \int\limits_\Omega |\Delta w|^{\frac{p}{p-1}}\, dx - \frac{|\lambda|^q}{q} \int\limits_\Omega |w|^q\, dx - \lambda \int\limits_\Omega hw\, dx,$$

and its restriction to the unit sphere $\mathbf{R} \times S$, that is

$$\tilde{I}_h(\lambda, w) = \frac{p-1}{p}|\lambda|^{\frac{p}{p-1}} - \frac{|\lambda|^q}{q} \int\limits_\Omega |w|^q\, dx - \lambda \int\limits_\Omega hw\, dx.$$

Now, we will prove that, if h is small enough, for any $w \in S$ the bifurcation equation

$$|\lambda|^{\frac{p}{p-1}-2}\lambda - |\lambda|^{q-2}\lambda \int\limits_\Omega |w|^q\, dx = \int\limits_\Omega hw\, dx \tag{4.1}$$

has at least three different roots $\lambda_i(w)$, $i = 1, 2, 3$. To this aim set

$$\psi_w(\lambda) = \frac{1}{\lambda}|\lambda|^{\frac{p}{p-1}}\left(1 - |\lambda|^{q-\frac{p}{p-1}} \int\limits_\Omega |w|^q\, dx\right).$$

From now on, let $1 < p \leqslant 2$ if $N = 2, 3$ or $1 < p < \frac{N}{N-2}$ if $N \geqslant 4$ and $q > 1 + \frac{1}{p-1}$ (small changes in the proof are needed in the case $\frac{N}{N-2} \leqslant p \leqslant 2$ and q such that $1 > \frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}$). By Lemma 2.2 it follows that

$$\lim_{\lambda \to 0^\pm} \psi_w(\lambda) = 0^\pm, \qquad \lim_{\lambda \to \pm\infty} \psi_w(\lambda) = \mp\infty,$$

hence, $\psi_w(\lambda)$ has a local maximum M_w and a local minimum m_w. Let us point out that, for a general nonlinearity f, we are not able to calculate M_w and m_w. On the contrary, if $f(u) = s^{q-1}$ the function $\psi_w(\lambda)$ is odd, $M_w = -m_w$ and direct calculations allow us to find M_w.

In fact, taken for simplicity $\lambda > 0$, we can write

$$\psi_w(\lambda) = \lambda^\alpha - c\lambda^\beta \quad \text{with } \alpha = \frac{1}{p-1},\ \beta = q-1,\ c = |w|_q^q.$$

Clearly,

$$\psi'_w(\lambda) = 0 \quad \text{if and only if} \quad \lambda = \left(\frac{\alpha}{c\beta}\right)^{\frac{1}{\beta-\alpha}},$$

hence,

$$M_w = -m_w = \left(\frac{\alpha}{c\beta}\right)^{\frac{\alpha}{\beta-\alpha}}\left(1 - \frac{\alpha}{\beta}\right),$$

that is, by the expressions of α, β and c, we obtain

$$M_w = -m_w = \left(\frac{1}{|w|_q^q (p-1)(q-1)}\right)^{\frac{1}{(p-1)(q-1)-1}} \left(1 - \frac{1}{(p-1)(q-1)}\right). \tag{4.2}$$

Then, Eq. (4.1) has three distinct roots if

$$\left|\int_\Omega hw\,dx\right| < M_w. \tag{4.3}$$

Let us remark that for all $w \in S$ it is

$$|w|_q \leqslant c_E|\Omega|\|w\|_E \leqslant c_E|\Omega|,$$

where c_E is the imbedding constant of E in $C(\Omega)$ and $|\Omega|$ denotes the Lebesgue measure of Ω, therefore by (4.2) we deduce

$$M_w \geqslant \left(\frac{1}{(c_E|\Omega|)^q (p-1)(q-1)}\right)^{\frac{1}{(p-1)(q-1)-1}} \left(1 - \frac{1}{(p-1)(q-1)}\right),$$

where the constant in the second member is independent of $w \in S$. Obviously, if we fix $h \in L^1(\Omega)$ with

$$\|h\|_1 < \frac{1}{c_E}\left(\frac{1}{(c_E|\Omega|)^q (p-1)(q-1)}\right)^{\frac{1}{(p-1)(q-1)-1}} \left(1 - \frac{1}{(p-1)(q-1)}\right),$$

then (4.3) holds and the bifurcation equation possesses three isolated smooth branches of solutions $\lambda_i = \lambda_i(w)$, $i = 1, 2, 3$, where $\lambda_1 < \lambda_2 \leqslant 0 < \lambda_3$ if $\int_\Omega hw\,dx \leqslant 0$ while $\lambda_1 < 0 < \lambda_2 < \lambda_3$ if $\int_\Omega hw\,dx > 0$. Moreover, by the implicit functions theory, $\lambda_i(w) \in C^1(S)$.

Hence, we obtain three distinct functionals

$$\begin{aligned}
\tilde{I}_{h,i}(w) &= \tilde{I}_{h,i}\big(\lambda_i(w), w\big) \\
&= \frac{p-1}{p}|\lambda_i|^{\frac{p}{p-1}} - \frac{|\lambda_i|^q}{q}\int_\Omega |w|^q\,dx - \lambda_i \int_\Omega hw\,dx \\
&= \left(\frac{p-1}{p} - \frac{1}{q}\right)|\lambda_i|^q \int_\Omega |w|^q\,dx - \frac{\lambda_i}{p}\int_\Omega hw\,dx
\end{aligned}$$

defined on $B - \{0\}$. We will prove that for each $i = 1, 2, 3$, $\tilde{I}_{h,i}$ attains its minimum at a point $\bar{w}_i \in S$ such that $\lambda_i(\bar{w}_i) \neq 0$. Indeed, arguing as in Section 2 it is easy to prove that $\tilde{I}_{h,i}$ is bounded from below. Moreover, we can extend Lemma 2.3 as follows.

Lemma 4.1. *If $\{w_n\} \subset S$ is a minimizing sequence of the functional $\tilde{I}_{h,i}$, then the corresponding sequence $\{\lambda_i(w_n)\}$ is bounded and bounded away from zero.*

Proof. Let $\{w_n\} \subset S$ such that $\tilde{I}_{h,i}(w_n) \to m_i = \inf_S \tilde{I}_{h,i}$. Clearly, up to subsequence, (2.8) holds.

For simplicity, suppose that the sequence $\{\lambda_n\} = \{\lambda_i(w_n)\}$ is positive. In order to prove the first part of (2.9), assume by contradiction that $\lambda_n \to 0$ (up to subsequence); consequently,

$$\tilde{I}_{h,i}(w_n) \to 0, \quad \text{i.e.} \quad m_i = 0. \tag{4.4}$$

Since $q > 1 + \frac{1}{p-1}$ and $\{|w_n|_q^q\}$ is bounded, (4.1) implies that

$$\int_\Omega h w_n \, dx > 0 \quad \text{for } n \text{ large}$$

and

$$\lim_n \int_\Omega h w_n \, dx = 0 \quad \text{with the same order of } (\lambda_n)^{\frac{1}{p-1}}.$$

Using again $q > 1 + \frac{1}{p-1}$, by the previous estimates and the expression of $\tilde{I}_{h,i}$ it follows that

$$\tilde{I}_{h,i}(w_n) < 0 \quad \text{for } n \text{ large},$$

and therefore $m_i < 0$, which contradicts (4.4).

Now, in order to state the second inequality in (2.9), assume by contradiction that $\lambda_n \to +\infty$ (passing to a subsequence). By the bifurcation equation, it is

$$(\lambda_n)^{\frac{1}{p-1}} \left(1 - (\lambda_n)^{q-1-\frac{1}{p-1}} |w_n|_q^q\right) = \int_\Omega h w_n \, dx, \tag{4.5}$$

therefore, the contradiction easily follows if $|w_n|_q^q \to |w|_q^q \neq 0$. On the other hand, if $w = 0$, we can consider

$$\lim_n (\lambda_n)^{q-1-\frac{1}{p-1}} |w_n|_q^q = l.$$

If $l \neq 1$, the contradiction follows again by (4.5). If $l = 1$, $\lambda_n^q |w_n|_q^q$ is an infinity equivalent to $(\lambda_n)^{1+\frac{1}{p-1}}$, then by the expression of $\tilde{I}_{h,i}$ we obtain that

$$\tilde{I}_{h,i}(w_n) \to +\infty,$$

which contradicts the fact that $\{w_n\}$ is a minimizing sequence. □

Arguing as in Section 2 we conclude that for all $i = 1, 2, 3$ the functional $\tilde{I}_{h,i}$ attains its minimum at a point $\bar{w}_i$ on S, then by the fibering method the original action functional I_h has at least three critical points of the form $\bar{u}_i(x) = \lambda_i \bar{w}_i(x)$ which are solutions of system (1.4). Finally, as the sign of $\lambda_i(\bar{w}_i)$ depends on the sign of $\int_\Omega h\bar{w}_i\,dx$, it is

$$\int_\Omega h\bar{u}_1\,dx \leqslant 0, \qquad \int_\Omega h\bar{u}_2\,dx \geqslant 0, \qquad \int_\Omega h\bar{u}_3\,dx \geqslant 0.$$

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